Math 8, Summer 2012
Exam 2

Short Ans.	
1	
2	
3	
Total	

Name______________________________

Perm No.____________________________

Directions:

1. Each problem is graded out of 4 points.
2. Each short answer question is worth 1 point.
3. You're only allowed a writing instrument and your wits.
4. Proofs should be clean, to the point, and written in proper English sentences.

Short Answer

1. Let S be a set with a relation R. Precisely define what it means for R to be reflexive.

2. Evaluate the sum $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$.

3. Dirichlet stuffs 316 pigeons into 21 holes. What's the largest number of pigeons we are guaranteed to see in a single hole?

4. A trichotomous relation $\simeq$ on a set S is one with the following property:

   For all $x, y \in S$, exactly one of the following is true: $x \simeq y$, $y \simeq x$, or $x = y$.

   Give an example of such a relation.

5. Given an equivalence relation $\sim$ on a set S and $x \in S$, define the equivalence class $[x]$.

6. Alice wants to prove $P(n)$ for all integers $n \geq 3$. She proves $\forall n \geq 3,\ P(n) \Rightarrow P(n+2)$. Which base cases are needed to complete the induction?

7. Which of these is an equivalence relation on $\mathbb{R}$?

   (a) $x \sim y \iff x - y \in \mathbb{Q}$
   (b) $x \sim y \iff x \leq y$
   (c) $x \sim y \iff x + y \in \mathbb{Z}$
   (d) $x \sim y \iff \sin x = \cos y$
   (e) None of the above

8. Give a precise statement of the well–ordering principle.

Problems

1. Prove for all integers $n \geq 0$
$$\int_0^\infty x^n e^{-x}\,dx = n!$$
Suggestion: Integrate by parts.

2. On a spherical surface, define a closed hemisphere to be a hemisphere which includes its edge as part of the set. Given 5 arbitrary points on the surface of a sphere prove there is a closed hemisphere containing 4 of them.

   Suggestion: It is easier to find a hemisphere that contains three of the points. Do this first. A slight adjustment to the proof finds one containing four of them.

3. Fix a positive integer n. Given an integer $k \geq 0$ define $d(k)$ to be the number of permutations of k objects leaving no objects in their starting positions. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} d(k) = n!$$

Suggestion: To permute n objects, you could first pick which objects move and which don't.